MATH 102:107, CLASS 5 (FRI SEPT 15)

(1) The function

$$f(x) = \frac{x^3 - ax^2}{x - a}$$

is undefined at $x = a$. What is $\lim_{x \to a} f(x)$?

(a) 0 (b) a (c) a^2 (d) a^3

Solution: The answer is (c). This is because

$$f(x) = \frac{x^3 - ax^2}{x - a} = \frac{x^2(x - a)}{x - a} = x^2$$

for any $x \neq a$. When $x = a$, the function is undefined - but the limit is computed by consider values of x *close* to a, but *not equal* to a. Therefore, the limit is equal to

$$\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2$$

(2) Consider the function

$$f(x) = \begin{cases} -1, & x < a \\ 1, & x \geq a \end{cases}$$

What is $\lim_{x \to a} f(x)$?

(a) 1 (b) -1 (c) 0 (d) DNE

Solution: The answer is (d). This is because the limit coming from the **left** is

$$\lim_{x \to a^-} f(x) = -1$$

and the limit coming from the **right** is

$$\lim_{x \to a^+} f(x) = 1$$

These are called the **one-sided limits**. Since they are not equal to each other, the two-sided limit $\lim_{x \to a} f(x)$ does not exist. (The function above is called a **piecewise-defined function**.)

(3) What is

$$\lim_{x \to a} \frac{1}{x - a}?$$

(a) ∞
(b) $-\infty$
(c) DNE
(d) More than one answer is correct

Solution: The answer is (c). As $x \to a^+$, the function tends to $+\infty$, while as $x \to a^-$, the function tends to $-\infty$. Technically, neither the right-hand limit nor the left-hand limit exists, and so the two-sided limit can't exist. (But even if we declared them to be $+\infty$ and $-\infty$, they would not be equal to each other, and even then the two-sided limit wouldn't exist.)

(4) Which of the following limits DNE?

(a) $\lim_{x\to 0} \frac{1}{x+1}$
(b) $\lim_{x\to -1} \frac{1}{x+1}$
(c) $\lim_{x\to -1} \frac{x+1}{x+1}$
(d) More than one DNE

Solution: The correct answer is (b). By the previous question, that limit does not exist - just set $a = -1$ in question 3. The other two limits are

$$\lim_{x\to 0} \frac{1}{x+1} = \frac{1}{0+1} = 1$$

by just plugging in $x = 0$, and

$$\lim_{x\to -1} \frac{x+1}{x+1} = \lim_{x\to -1} 1 = 1$$

by the same argument as in question 1.

(5) What value of a makes the following function continuous?

$$f(x) = \begin{cases} ax^2 + 1, & x < 1 \\ -x^3 + x, & x \geq 1 \end{cases}$$

(a) $a = 1$
(b) $a = 0$
(c) $a = -1$
(d) No value of a works

Solution: The correct answer is (c). The piecewise-defined function is continuous everywhere except possible at the value $x = 1$. It's continuous if and only if the two pieces 'match up' at that value, i.e. **only if** $ax^2 + 1$ **and** $-x^3 + x$ **are equal to each other at** $x = 1$**.** Plugging $x = 1$ into both of these expressions,

$$a(1)^2 + 1 = -(1)^3 + 1 \iff a + 1 = -1 + 1 \iff a = -1$$

So $a = -1$ causes the function $f(x)$ to be continuous (while any other value of a will cause it to have a discontinuity at $x = 1$).

(6) Which of the following limits exist?

(a) $\lim_{x\to\infty} x^3$ (b) $\lim_{x\to\infty} x^{\frac{1}{3}}$ (c) $\lim_{x\to\infty} 3^x$ (d) $\lim_{x\to\infty} x^{-3}$

Solution: Only (d) exists. As $x \to \infty$, $x^3 \to \infty$, $x^{\frac{1}{3}} = \sqrt[3]{x} \to \infty$, $3^x \to \infty$, and $x^{-3} = \frac{1}{x^3} \to 0$.

(7) What is the following limit?

$$\lim_{x\to\infty} \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}{6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1}$$

(a) 0 (b) 6 (c) $\frac{1}{6}$ (d) DNE

Solution: As $x \to \infty$, the numerator and denominator can be asymptotically simplified to just the term of highest degree

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6 \sim x^5$$

$$6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1 \sim 6x^5$$

Therefore

$$\lim_{x\to\infty} \frac{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}{6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1} = \lim_{x\to\infty} \frac{x^5}{6x^5} = \lim_{x\to\infty} \frac{1}{6} = \frac{1}{6}$$

(8) To compute the derivative of the function $f(x) = \frac{1}{x+2}$, we need to compute:

(a) $\lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$

(b) $\lim_{x\to 1} \frac{\frac{1}{x+2} - \frac{1}{x}}{x}$

(c) $\lim_{h\to 1} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$

(d) $\lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{x}$

Solution: The correct answer is (a). See the definition of the derivative in the notes. The expression inside the limit, i.e. $\frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$, is the average rate of change over the interval $[x, x+h]$, or equivalently, the slope of the **secant line** defined by this interval.

(9) Calculate the derivative of the function $f(x) = x^3 - x + 1$, and use this to write an equation for the tangent line at $x = 0$.

Solution: Many of you know how to use the Power Rule to shortcut this derivative computation and say $f'(x) = 3x^2 - 1$, but I will go ahead and explicitly compute the derivative of x^3 here to show how this goes.

$$\text{Derivative of } x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$= \lim_{x\to 0}(3x^2+3xh+h^2) = 3x^2$$

The derivative of the function $-x$ is -1 (because its graph is a line with slope -1 everywhere), while the derivative of the function 1 is 0 (because it's a horizontal line with no slope). Hence, the sum $f(x) = x^3 - x + 1$ has derivative $f'(x) = 3x^2 - 1$.

Plugging in $x = 0$, we get $f'(0) = 3(0)^2 - 1 = -1$. So the tangent line at $x = 0$ has

(a) Slope -1.
(b) Passes through the point $(0, f(0)) = (0, 1)$.

A line of slope -1 and passing through $(0, 1)$, has equation $y - 1 = -1(x - 0)$, by the point-slope formula. Cleaning this up a bit, we get the equation $\boxed{y = -x + 1}$. (Note: this is what we meant in Class 2 when we said $x^3 - x$ *looks like* $-x$ near $x = 0$. That is the tangent line at $x = 0$!)

(10) (Bonus Challenge) Use the definition of the derivative and the following hint to calculate the derivative of the function $f(x) = \sqrt{x}$. Hint:

$$\frac{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

Solution: Setting up the limit to calculate the derivative,

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Now, using the hint, we have that

$$\sqrt{x+h}-\sqrt{x} = \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}} = \frac{h}{\sqrt{x+h}+\sqrt{x}}$$

and thus,

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\frac{h}{\sqrt{x+h}+\sqrt{x}}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}=\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

is the derivative of $f(x) = \sqrt{x}$.